Lecture 1

Outline

1. Probability space
   - Sample space
     - Elementary outcomes
     - Events
   - Probability function
     - Axioms
   - Properites of a probability function

2. Examples of probability spaces
   - Classical definition
   - Discrete sample spaces
   - Uncountably infinite sample space
     - Geometrical probability

Instructor

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Structure of the course

- Lectures
  - presenting the content of the course (definitions, theorems, some proofs, examples)
  - the PDF files of the slides will be posted in advance of each lecture so you will be able to print them out and use them for taking notes
  - the files will not contain all the material that will actually be presented in the class
- Recitations/tutorials
  - solving mathematical problems related to the content of the course

Evaluation

Your final grade will be based on:

- two tests (with 25 points each)
  - held on during the semester
  - solving mathematical problems similar to the ones from the problem sets
  - *formulas.pdf* and *tables.pdf* from my web page are the only documents allowed on the test
  - no calculators and/or other electronic devices are allowed on the test
- a final exam (50 points)
  - to be held in the exam session, two dates in June and one in September (the final exams dates and locations will be available at a later date on your department's website, and will be announced when it is posted)
  - *formulas.pdf* and *tables.pdf* from my web page are the only documents allowed on exams
  - no calculators and/or other electronic devices are allowed on exams
- activity points (10 points - additional)
  - solving problems form the problem sets during the classes,

Evaluation - cont'd

- Concluding:
  - two tests (50 points)
  - a final exam (50 points)
  - activity (additional 10 points)
- In order to pass the course, you need to receive at least 50 points (two test + final exam + activity)
- The final grade will be assigned as follows:
  - < 50 – 2.0
  - $50 - 60$ – 3.0
  - $60 - 70$ – 3.5
  - $70 - 80$ – 4.0
  - $80 - 90$ – 4.5
  - $90 - 100$ – 5.0

Probability space

Definition (Sample space, Ω)

The **sample space** Ω of an experiment is the set of all possible outcomes (**elementary outcomes**, **sample points**) of the experiment.

Definition (Event)

An **event** A is a subset of the sample space Ω and we say that A occurred if the actual outcome is in A.

Example (Coin flip)

A coin is flipped once.

- There are two possible outcomes, heads (H) and tails (T).
- The sample space is $\Omega = \{H, T\}$.
- The events are: $\{H, T\}$, $\{H\}$, $\{T\}$, $\emptyset$.

Example

Roll a six-sided die. Let $A =$ "the even number faces up on the die", so $A = \{2, 4, 6\}$.

- If the die results in "2", then A occurs, since $2 \in \{2, 4, 6\}$.
- If the die results in "3", then A does not occur, since $2 \notin \{2, 4, 6\}$.

Example (Coin flips)

A coin is flipped 10 times.

- Any elementary outcome is a 10-long sequence of heads and tails (e.g., $HHHHTTHTTT$).
- The sample space is the set of all possible sequences od length 10 of $H's$ and $T's$: $\Omega = \{s_1 s_2 \ldots s_{10} : s_i \in \{H, T\} \;\; for \;\; i = 1, \ldots, 10\}$.
- Examples of events:
  - Let $A_j =$" jth flip is heads":

    $$A_j = \{s_1 \ldots s_{j-1} H s_{j+1} \ldots s_{10} : s_i \in \{H, T\}, i \neq j\},$$

    A_j is an event, since $A_j \subset \Omega$.
  - Let B="at least one flip is heads",

    $$B = \bigcup_{j=1}^{10} A_j \subset \Omega$$

Example (Waiting for the first head)

A coin is flipped until the first H comes up.

- An elementary outcome is a sequence of $T's$ and H at the end.
- The sample space Ω:

$$\Omega = \{H, TH, TTH, TTTH, TTTTH, \ldots\},$$

  (Ω is countably infinite.)
- Examples of events:
  - Let $A =$ "there were at least four flips (until H comes up)",

$$A = \{TTTH, TTTTH, \ldots, \}$$

  - Let B="there was an odd number of flips",

$$B = \{H, TTH, TTTTH, \ldots\}$$

Example (Picking a point from an interval)

Pick a random point from interval $[0, 1]$. The sample space Ω is the set of all points that belong to $[0, 1]$. (Ω is uncountably infinite.)
Examples of events:

- $[0.2, 0.785]$,
- $(0.1, 0.5) \cup (0.8, 0.9]$.

Definition (Probability function, $\mathbb{P}$)

A **probability function** $\mathbb{P}$ assigns a number $\mathbb{P}(A) \in [0,1]$, called the **probability** of A, to every event A. The function $\mathbb{P}$ must satisfy the following **axioms**:

1. $\mathbb{P}(\Omega) = 1$,
2. If $A_1, A_2, \ldots,$ are disjoint events, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i),$$

- disjoint events means that they are mutually exclusive: $A_i \cap A_j = \emptyset \;\; \textit{for} \;\; i \neq j$.

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Theorem (Properties of probability)

Probability has the following properties, for any events A and B:

1. $\mathbb{P}(\emptyset) = 0.$
2. $\mathbb{P}(A^c) = 1 - \mathbb{P}(A).$
3. *If* $A \subseteq B$, *then* $\mathbb{P}(A) \leq \mathbb{P}(B).$
4. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$

Examples of probability spaces

Classical definition of probability

Let Ω be a finite set, e.g., $\Omega = \{\omega_1, \dots, \omega_n\}$. Assuming that

$$\mathbb{P}\left(\{\omega_i\}\right) = \frac{1}{n}, \quad \forall i = 1, 2 \dots, n,$$

we use the classical definition of probability.

Definition (Classical definition)

Let A be an event for an experiment with a finite sample space Ω. The **classical probability** of A is

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{\text{number of outcomes favorable to A}}{\text{total number of outcomes in } \Omega}.$$

For any $A \subset \Omega$

$$\mathbb{P}(A) = \sum_{i:\omega_i \in A} \mathbb{P}(\{\omega_i\}) = \sum_{i:\omega_i \in A} \frac{1}{n} = \frac{\#\{i : \omega_i \in A\}}{n} = \frac{|A|}{|\Omega|}.$$

Example

Flip a fair coin once. Since the coin is fair, we should assign equal probabilities to $\{H\}$ and $\{T\}$: $\mathbb{P}(\{H\}) = \mathbb{P}(\{T\}) = \frac{1}{2}$. Therefore,

$$\mathbb{P}(\{H, T\}) = 1, \;\; \mathbb{P}(\{H\}) = \mathbb{P}(\{T\}) = \frac{1}{2}, \quad \mathbb{P}(\emptyset) = 0,$$

and $\mathbb{P}$ satisfies the axioms.

Example

Flip a fair coin ten times.

$$\Omega = \{s_1 s_2 \ldots s_{10} : s_i \in \{H, T\}, i = 1, 2, \ldots, 10\}, so \;\; |\Omega| = 2^{10},$$

$$\mathbb{P}(\{s_1 s_2 \ldots s_{10}\}) = \frac{1}{2^{10}} \;\; \text{(all sample points are "equaly likely").}$$

Let A="there is one H", then

$$A = \{HT \ldots T, TH \ldots T, \ldots, TT \ldots TH\} \subset \Omega,$$

$|A| = 10$, so $\mathbb{P}(A) = \frac{10}{2^{10}}$.

Discrete sample spaces

Generalizing the classical definition we can specify the probability function for discrete (finite or countably infinite) sample spaces.

Let Ω be a countable set, $\Omega = \{\omega_1, \omega_2, \ldots\}$. We define $\mathbb{P}$ in the following way:

$$\mathbb{P}(\{\omega_i\}) = p_i, \quad \text{where} \quad p_i \in (0,1), i = 1, 2, \ldots \quad \text{and} \quad \sum_i p_i = 1.$$

Then for any event $A \subset \Omega$:

$$\mathbb{P}(A) := \sum_{i:\omega_i \in A} p_i.$$

Such $\mathbb{P}$ is a valid probability function (it satisfies probability axioms).

Remark

If the sample space consists of a countable number of possible outcomes (sample points), then the probability function is specified by the probabilities of the sample points (consisting of a single element).

Example

Flip a fair coin until H comes up. What is the probability that we will need an even number of flips? Is it $\frac{1}{2}$?

Here $\Omega = \{H, TH, TTH, \ldots\}$ is a countable set,. Let us define $\mathbb{P}$:

$$\mathbb{P}(\{H\}) = \frac{1}{2},\ \mathbb{P}(\{TH\}) = \frac{1}{4}, \mathbb{P}(\{TTH\}) = \frac{1}{8}, \ldots,$$

general rule:

$$\mathbb{P}(\{\underbrace{T\ldots T}_{n-1}H\}) = \frac{1}{2^n}.$$

Let A="there was an even number of flips",

$$A = \{TH\} \cup \{TTTH\} \cup \{TTTTTH\} \cup \ldots,$$

so

$$\mathbb{P}(A) = \mathbb{P}(\{TH\}) + \mathbb{P}(\{TTTH\}) + \ldots = \sum_{k=1}^{\infty} \frac{1}{2^{2k}} = \frac{1}{3}.$$

Uncountably infinite sample space

Example

Pick a random point from an interval $[0, 1]$. What is the probability of A that the chosen point belongs to $\left[\frac{1}{4}, \frac{3}{4}\right]$?

- the possible outcomes are the numbers in $[0, 1]$, so $\Omega = [0, 1]$,
- intuitively all the outcomes are "equally likely",
- but, we cannot assign positive probability for a single outcome

Assigning positive probabilities to the single outcomes would imply that events with sufficiently large number of elements would have probability larger than 1!
Therefore probability of any elementary outcome must be 0.

Example (Cont'd)

Q.:How can we define probability function on $\Omega = [0, 1]$?
A.: Assign probability $b - a$ to any subinterval $[a, b]$ of $[0, 1]$.

This assignment defines a valid probability function, since it satisfies all probability axioms.
Solution: $A = \left[\frac{1}{4}, \frac{3}{4}\right]$, so $\mathbb{P}(A) = |\frac{3}{4} - \frac{1}{4}| = \frac{1}{2}$.

Geometrical probability

Let $\Omega \subset \mathbb{R}^n$ be a set with a finite measure (length for $n = 1$, area for $n = 2$, volume for $n = 3$).
For any event $A \subset \Omega$,

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|},$$

defines **the geometrical probability**.

Example

Pick two points at random from $[0, 1]$. What is the probability of $A =$ "the sum of two chosen points (numbers) is at most $\frac{1}{4}$"?

$\Omega = [0, 1] \times [0, 1]$ and $A = \{(x, y) \in \Omega : x + y \leq \frac{1}{4}\}$.
The area of A: $|A| = \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{32} \implies \mathbb{P}(A) = \frac{1}{32}$.