Lecture 2

Outline

1. Conditional probability
    - The chain rule
    - Law of total probability
    - Bayes' rule
2. Independence of events

Conditional probability

Example

- Roll a fair six-sided die. What is the probability that "4" comes up?
- Given that the even number faces up on the die, what is the probability that "4" comes up?
- Given that the odd number faces up on the die, what is the probability that "4" comes up?

Definition

Let $(\Omega, \mathbb{P})$ be a probability space. For any events A and B such that $\mathbb{P}(B) > 0$, **the conditional probability** of event A given B is defined as

$$\mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

The conditional probability measures the probability of A given that we know that B occurred.

Example (Cont'd)

$\Omega = \{1,2,3,4,5,6\},\ \mathbb{P}(\{i\}) = \frac{1}{6},\ i = 1,2,\ldots,6$. Let A=„die faces up "4"", then

- $\mathbb{P}(A) = \frac{\overline{\overline{A}}}{\overline{\overline{\Omega}}} = \frac{1}{6}$,
- B - „even number comes up", $\mathbb{P}(B) = \frac{1}{2}$, $\mathbb{P}(A|B) = \frac{1}{3}$
- C - „odd number comes up", $\mathbb{P}(C) = \frac{1}{2}$, $\mathbb{P}(A|C) = 0$.

Example

Toss a fair coin three times. Find $\mathbb{P}(A|B)$ when A and B are the events:

$$A = \{\textit{more Hs than Ts come up}\}, \quad B = \{\textit{at least one toss is a H}\}.$$

Solution

$$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\},$$

$$B = \{HHH, HHT, HTH, THH, HTT, THT, TTH\}$$

$$A \cap B = \{HHH, HHT, HTH, THH\},$$

Therefore $\mathbb{P}(B) = \frac{7}{8}$, $\mathbb{P}(A \cap B) = \frac{4}{8}$, *so* $\mathbb{P}(A|B) = \frac{4}{7}$.

Example

A fair die with four faces is rolled twice . Find $\mathbb{P}(A_i|B)$, where

$$A_i = \{i \text{ is the maximum of two rolls}\}, \quad i = 1, \ldots, 4.$$

$$B = \{\text{minimum of two rolls is } 3\},$$

Solution

$$\mathbb{P}(A_i|B) = \begin{cases} 0, & i \leq 3, \\ \frac{1}{3}, & i = 3, \\ \frac{2}{3}, & i = 4. \end{cases}$$

Conditional probabilities are probabilities

$(\Omega, \mathbb{P})$ - probability space. Fix an event B with $\mathbb{P}(B) > 0$.
For any event A, define

$$\mathbb{P}_B(A) := \mathbb{P}(A|B).$$

Then function $\mathbb{P}_B$ satisfies:

1. $\mathbb{P}_B(\Omega) = 1$,
2. If $A_1, A_2, \ldots$, are disjoint events, then $\mathbb{P}_B(U_n A_n) = \sum_n \mathbb{P}_B(A_n)$.

So $\mathbb{P}_B$ satisfies the axioms of probability.

<u>Conclusion</u>: If we are told that an event B has occurred in the experiment, then B replaces the entire sample space Ω (as a new set of possible outcomes) and consequently the probability of any event changes.

The chain rule

Theorem

For any events $A_1, \ldots, A_n$ *satisfying*

$$\mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_{n-1}) > 0,$$

the following formula holds:

$$\mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2|A_1) \cdots \mathbb{P}(A_n|A_1 \cap A_2 \cap \ldots \cap A_{n-1}).$$

Example

An urn contains 4 white balls and 5 black balls. We randomly draw 3 balls from the urn (without replacement). What is the probability that all three balls are black?

Law of total probability

Definition (Countable partition of Ω)

$(\Omega, \mathbb{P})$ - probability space. The family of events $(H_n)_n$ is a **countable partition of** Ω, if

1. $\mathbb{P}(H_n) > 0 \;\; \forall n$,
2. $\bigcup_n H_n = \Omega$,
3. $H_i \cap H_j = \emptyset$ for $i \neq j$.

Example

Roll a six-sided die. Let $A_i =$ „die faces up i spots", $i = 1, 2, \ldots, 6$.

- $\mathbb{P}(H_i) = \frac{1}{6}$, $i = 1, 2, \ldots, 6$,
- $\bigcup_{i=1}^{6} A_i = \Omega$,
- $A_i \cap A_i = \emptyset$, $i \neq j$,

$\implies$ Events: $A_1, A_2, \ldots, A_6$ comprise the finite partition of Ω.

Theorem (Law of total probability)

$(\Omega, \mathbb{P})$ - *probability space,* $(A_n)_n$ *- countable partition of* Ω*. Then* $\forall A$*:*

$$\mathbb{P}(A) = \sum_n \mathbb{P}(A|A_n)\mathbb{P}(A_n).$$

Example

You have one fair coin and one biased coin (with two Heads). You pick one of the coins at random and flip it three times. What is the probability that the coin lands Heads all three times?

Example

There are two urns. Urn I contains 2 white balls and 3 black balls and urn II contains 4 white and 1 black ball. Firstly, we pick one urn at random, and then we draw one ball out of it. What is the probability of choosing a black ball?

Theorem (Bayes' rule)

$(\Omega, \mathbb{P})$ - *probability space,* $(A_n)_n$ - *countable partition of* Ω*. Let* A *be an event such that* $\mathbb{P}(A) > 0$*, then*

$$\mathbb{P}(A_n|A) = \frac{\mathbb{P}(A|A_n)\mathbb{P}(A_n)}{\mathbb{P}(A)}.$$

Example

You have one fair coin and one biased coin (with two Heads). You pick one of the coins at random and flip it three times. It lands Heads all three times. Given this information, what is the probability that the coin you picked is the fair one?

Example

There are two urns. Urn I contains 2 white balls and 3 black balls and urn II contains 4 white and 1 black ball. Firstly, we pick one urn at random, and then we draw one ball out of it. It turns out to be white. Given this information, what is the probability that the ball was selected from II urn?

Independence of events

Independence (of events) means that events do not provide any information about each other.

Example

Flip two coins, A - „first coin comes up H", B - „second coin comes up T". $\mathbb{P}(A) = ?$, $\mathbb{P}(A|B) = ?$

Definition

Events A and B are **independent** if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

If $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$, then this is equivalent to

$$\mathbb{P}(A|B) = \mathbb{P}(A),$$

and also equivalent to $\mathbb{P}(B|A) = \mathbb{P}(B)$.