Lecture 5

Outline

1. Discrete distributions
   - Degenerate distribution
   - Two point distribution
     - Bernoulli distribution
   - Binomial distribution
   - Geometric distribution
   - Poisson distribution

2. Continuous distributions
   - Uniform distribution
   - Exponential distribution
   - Normal distribution
     - Standard normal distribution

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A survey of probability distributions

We specify distribution of a random variable in a different way according to their type:

- probability mass function (PMF) - discrete case,
- probability density function (PDF) - continuous case,
- cumulative distribution function (CDF) - used in both cases.

Degenerate distribution

One point distribution. There exists $a \in \mathbb{R}$ such that $S_X = \{a\}$, so $\mathbb{P}(X = a) = 1$. The corresponding cumulative distribution function:

$$F_X(t) = \begin{cases} 0, & t < a, \\ 1, & t \geq a. \end{cases}$$

Two point distribution

$S_X = \{x_1, x_2\}$, $\mathbb{P}(X = x_1) = p \in (0,1)$ and $\mathbb{P}(X = x_2) = 1 - p$.
If $x_1 < x_2$, then

$$F_X(t) = \begin{cases} 0, & t < x_1, \\ p, & t \in [x_1, x_2), \\ 1, & t \geq x_2. \end{cases}$$

Bernoulli distribution, $X \sim B(p)$, $p \in (0,1)$
$S_X = \{0, 1\}$,

$$\mathbb{P}(X = 1) = p = 1 - \mathbb{P}(X = 0).$$

Example

Consider the coin tossing experiment, for which H comes up with probability p and T with probability $1 - p$. Let $X(\{H\}) = 1$ and $X(\{T\}) = 0$. Its PMF is

$$p_X(1) = p, \; p_X(0) = 1 - p.$$

The Bernoulli random variable is used to model probabilistic situations with just two outcomes.

Binomial distribution, $X \sim b(n, p)$

A r.v. X is said to have a binomial distribution with parameters n and p, if $S_X = \{0, 1, 2, \ldots, n\}$ and

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \ldots, n.$$

X can be interpreted the number of successes among n independent Bernoulli trials (each trial can result in one of two possible outcomes: success(S) or failure(F)).

One can verify that p_X is a valid PMF:

$$\sum_{k \in S_x} p_X(k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + (1-p))^n = 1.$$

Observe that

- the Bernoulli distribution is the special case of the binomial distribution: $B(p)$ is the same as $b(1, p)$.
- $X \sim b(n, p)$ can be expressed as

$$X = \sum_{i=1}^{n} X_i,$$

  where $X_i \sim B(p)$, $i = 1, 2, \ldots, n$.

Geometric distribution

A r.v. X has geometric distribution with parameter p, $X \sim g(p)$, if $S_X = \{1, 2, 3, \ldots\}$ and

$$p_X(k) = (1 - p)^{k-1} p, \quad k = 1, 2, \ldots .$$

X can be interpreted as the waiting time for the first success in independent Bernoulli trials with probability of success equals p.

Verify that $\sum_k p_X(k) = 1$.

Applications

- reliability theory (lifetime of the device - the time for the first break down)

Geometric tail distribution

If $X \sim g(p)$, then $\mathbb{P}(X > k) = (1-p)^k$, $k = 0, 1, 2, \ldots$.

Proof.

$\mathbb{P}(X > k) = \mathbb{P}(\{\text{no success in the first k trials}\}) = (1-p)^k$.

You can compute it directly (in more tedious way):

$$\mathbb{P}(X > k)$$

$$= \sum_{j=k+1}^{\infty} (1-p)^{j-1} p = (1-p)^k p \sum_{j=0}^{\infty} (1-p)^j = \frac{(1-p)^k p}{1-(1-p)} = (1-p)^k.$$

□

Theorem (Lack of memory property)

If $X \sim g(p)$, $\forall m, n \in \mathbb{N}$ *then*

$$\mathbb{P}(X > m + n | X > n) = \mathbb{P}(X > m).$$

Proof.

$$\mathbb{P}(X > m + n | X > n) = \frac{\mathbb{P}(\{X > m + n\} \cap \{X > n\})}{\mathbb{P}(X > n)} = \frac{\mathbb{P}(X > m + n)}{\mathbb{P}(X > n)}$$

□

Poisson distribution

A r.v. X has Poisson distribution with parameter λ, $X \sim \mathcal{P}(\lambda)$, if $S_X = \mathbb{N} \cup \{0\}$ and

$$p_X(k) = \mathbb{P}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

Verfify that $\sum_k p_X(k) = 1$.

The Poisson random variable describes the situation in which we deal with a **very large** number of independent repetitions of a Bernoulli trial (n) having a **very small** probability of success (p):

Remark

If $X \sim b(n, p)$, with n large and p small, then

$$p_X(k) \approx e^{-np} \frac{(np)^k}{k!},$$

i.e. X is distributed approximately the same as a $\mathcal{P}(\lambda)$, where $\lambda = np$.

Example

Only 0.5% of people activate an airport metal detector. Let X be the number of people out of 500 who activate the detector. Using the Poisson approximation compute:

- $\mathbb{P}(X = 5)$,
- $\mathbb{P}(X \geq 3)$.

Solution

- $$\mathbb{P}(X = 5) = \binom{500}{5} \left(\frac{5}{1000}\right)^5 \left(\frac{995}{1000}\right)^{495} \approx e^{-\lambda} \frac{\lambda^5}{5!},$$
- $$\begin{aligned} \mathbb{P}(X \geq 3) = 1 - \mathbb{P}(X < 3) = 1 - \mathbb{P}(X = 0) - \mathbb{P}(X = 1) - \mathbb{P}(X = 2) \\ \approx 1 - e^{-\lambda} - e^{-\lambda}\lambda - e^{-\lambda}\frac{\lambda^2}{2!}, \end{aligned}$$

  where $\lambda = \frac{5}{1000} \cdot 500 = 2.5$.

Uniform distribution (Continuous distributions)

A random variable X is said to be **uniform** on the interval $[a, b]$, $X \sim \mathcal{U}[a, b]$ if its pdf is of the form:

$$f(x) = \begin{cases} \frac{1}{b-a}, & x \in [a, b], \\ 0, & otherwise. \end{cases}$$

The density formula yields $\int_{\mathbb{R}} f(x)dx = 1$. The support of X: $S_X = [a, b]$ and the cumulative distribution function is of the form:

$$F(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & x \in [a, b), \\ 1, & x \geq b. \end{cases}$$

Exponential distribution

A random variable X is said to have **exponential distribution**, if its pdf is of the form

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & otherwise, \end{cases}$$

$\lambda > 0$ is called the rate of the distribution, $X \sim Exp(\lambda)$.

Theorem (The Memoryless Property)

$$\mathbb{P}(X > t + s | X > s) = \mathbb{P}(X > t), \quad \textit{for any } s, t > 0$$

Proof.

$$\mathbb{P}(X > t + s | X > s) = \frac{\mathbb{P}(X > t + s)}{\mathbb{P}(X > s)} = \frac{\int_{t+s}^{\infty} \lambda e^{-\lambda x} dx}{\int_{s}^{\infty} \lambda e^{-\lambda x} dx} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} =$$

$$= e^{-\lambda t} = \mathbb{P}(X > t).$$

□

Example

A study of the response time of a certain computer system yields that the response time in seconds has an exponentially distributed time with parameter 0.25. What is the probability that the response time exceeds 5 seconds?

Solution

X-r.v. denoting the response time, $X \sim Exp(0.25)$.

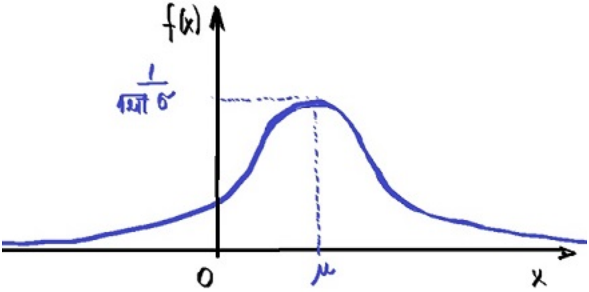
$$\mathbb{P}(X > 5) = \int_5^\infty 0.25e^{-0.25x}dx = -e^{-0.25x}\Big|_5^\infty = e^{-0.25\cdot 5} = e^{-1.25}.$$

Normal distribution

A random variable X is said to be **normal**, $X \sim \mathcal{N}(\mu, \sigma^2)$, if its pdf is of the form

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

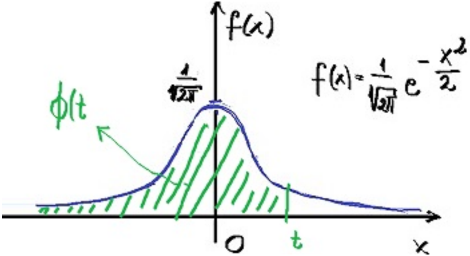
$\mu \in \mathbb{R}$, $\sigma > 0$ - two parameters.



The normalization property holds for f_X: $\int_{-\infty}^{\infty} f_X(x)dx = 1$.

Standard normal distribution

Here $\mu = 0$ and $\sigma = 1$.

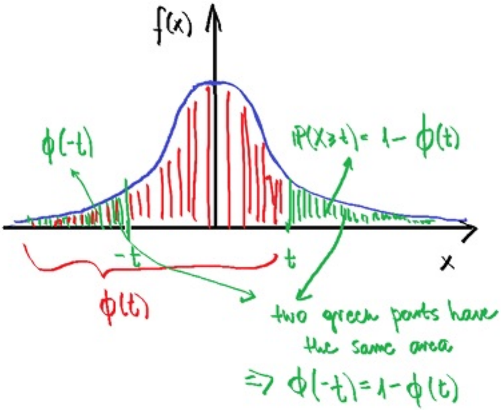


The corresponding cumulative distribution function is denoted by Φ:

$$\Phi(t) = \mathbb{P}(X \leq t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$

From the symmetry of the pdf of $\mathcal{N}(0,1)$, we can derive the following formula:

$$\Phi(-t) = 1 - \Phi(t).$$



The values of Φ are recorded in a special table. It allows us to calculate probabilities involving normal random variables.

Proposition

Let $X \sim \mathcal{N}(\mu, \sigma^2)$*, then*

$$F_X(t) = \Phi\left(\frac{t-\mu}{\sigma}\right)$$

Proof.

$$F_X(t) = \mathbb{P}(X \le t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Using u-Subsitiution such that $u = \frac{x-\mu}{\sigma}$, we get $dx = \sigma du$ and

$$\int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \int_{-\infty}^{(t-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du = \Phi\left(\frac{t-\mu}{\sigma}\right)$$

□

Example

Let $X \sim \mathcal{N}(2, 9)$. Compute $\mathbb{P}(X \leq 5)$ and $\mathbb{P}(-1 \leq X \leq 3)$.

Solution

$$\mathbb{P}(X \leq 5) = F_X(5) = \Phi\left(\frac{5-2}{3}\right) = \Phi(1) = 0,84$$

$$\mathbb{P}(-1 \leq X \leq 3) = F_X(3) - F_X(-1) = \Phi\left(\frac{3-2}{3}\right) - \Phi\left(\frac{-1-2}{3}\right)$$

$$= \Phi\left(\frac{1}{3}\right) - \Phi(-1) = \Phi\left(\frac{1}{3}\right) - (1 - \Phi(1)) \approx 0.63 - 1 + 0.84 = 0.47.$$