Lecture 6

Outline

1. Expected value - discrete case
   - Properties of expected values
2. Variance and standard deviation

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Numerical characteristics of random variables

Discrete case

Definition (Expected value)

Let X be a random variable taking the values in a discrete set S_X. The **expected value** (**expectation**) of the random variable X, denoted by $\mathbb{E}(X)$, is defined as

$$\mathbb{E}(X) = \sum_{x \in S_X} x p_X(x).$$

This is well-defined so long as $\sum_{x \in S_X} |x| p_X(x)$ converges.

Remark (How do we interpret the expected value?)

- *the average obtained after many trials (when you interpret the probabilities as the frequencies),*
- *if we place an object of mass $p_X(x)$ at position x for each $x \in S_X$, then $\mathbb{E}(X)$ is the position of* ***the center of gravity****.*

Example

Random variable X has the following distribution:

1. $\mathbb{P}(X=-1)=0.2,\ \mathbb{P}(X=0)=0.4,\ \mathbb{P}(X=1)=0.4,\ \mathbb{E}(X)=?$
2. $\mathbb{P}(X=-1)=0.4,\ \mathbb{P}(X=0)=0.2,\ \mathbb{P}(X=1)=0.4,\ \mathbb{E}(X)=?$
3. $\mathbb{P}(X=-1)=0.2,\ \mathbb{P}(X=0)=0.4,\ \mathbb{P}(X=100)=0.4,\ \mathbb{E}(X)=?$
4. Assume that $S_X=\{0,1,2,\ldots,n\}$, $\mathbb{P}(X=0)=\mathbb{P}(X=1)=\ldots\mathbb{P}(X=n)=\frac{1}{n+1}$. What is the center of gravity for this structure?

Solution

1. $\mathbb{E}(X)=(-1)\cdot 0.2+0\cdot 0.4+1\cdot 0.4=0.2,$
2. $\mathbb{E}(X)=(-1)\cdot 0.4+0\cdot 0.2+1\cdot 0.4=0,$
3. $\mathbb{E}(X)=(-1)\cdot 0.2+0\cdot 0.4+100\cdot 0.4=39.8,$
4. $\mathbb{E}(X)=\frac{1}{n+1}\left(0+1+2+\ldots+n\right)=\frac{1}{n+1}\cdot\frac{n(n+1)}{2}=\frac{n}{2}.$

Remark

- *The expected value is a summary statistic, providing a measure of the location of a random variable.*
- *The expected value is also called the mean or average of X (often denoted by μ).*
- *The expected value is a weighted average of the possible values (with probabilities as weights).*
- *If all the values are equally probable then the expected value is just the usual average of the values.*

Algebraic properties of $\mathbb{E}(X)$

Let X and Y be random variables defined on the same probability space Ω, then

1. $$\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y),$$
2. $$\forall a, b \in \mathbb{R} \quad \mathbb{E}(aX+b) = a\mathbb{E}(X) + b,$$
3. if X takes on only nonnegative values, then $\mathbb{E}(X) \geq 0$.

Example

Roll two dice, and let X be the sum of the faces. Find $\mathbb{E}(X)$.

Solution

$X = X_1 + X_2$, *where* X_i *- the result on the "i"th die,*
$\mathbb{E}(X) = \mathbb{E}(X_1 + X_2) = \mathbb{E}(X_1) + \mathbb{E}(X_2)$
Since X_1 *and* X_2 *have the same distributions,* $\mathbb{E}(X_1) = \mathbb{E}(X_2)$ *and*

$$\mathbb{E}(X_1) = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \ldots \frac{1}{6} \cdot 6 = 3.5,$$

so $\mathbb{E}(X) = 7$.

$\mathbb{E}X$ of Bernoulli distribution

$X \sim B(p)$,
PMF of X: $p_X(1) = p$ and $p_X(0) = 1 - p$.

$$\mathbb{E}(X) = 1 \cdot p + 0 \cdot (1 - p) = p.$$

Example

Let A be an event and

$$X = \begin{cases} 1, & \textit{if } A \textit{ occurs}, \\ 0, & \textit{otherwise}. \end{cases}$$

Compute $\mathbb{E}(X)$.

Solution

PMF of X:
$p_X(1) = \mathbb{P}(A)$ *and* $p_X(0) = \mathbb{P}(A^c) = 1 - \mathbb{P}(A)$, *so*
$\mathbb{E}(X) = 1 \cdot \mathbb{P}(A) + 0 \cdot (1 - \mathbb{P}(A)) = \mathbb{P}(A)$.

$\mathbb{E}X$ of Binomial distribution

$X \sim b(n,p)$,
PMF of X:

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \dots, n.$$

- $\mathbb{E}(X)$ can be computed directly as

$$\mathbb{E}(X) = \sum_k k p_X(k) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k} = np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} = np \sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} = np.$$

- Second approach: $X = \sum_{i=1}^{n} X_i$, where $X_i \sim B(p)$,

$$\mathbb{E}(X) = \sum_{i=1}^{n} \mathbb{E}(X_i) = \sum_{i=1}^{n} p = np.$$

$\mathbb{E}(X)$ of geometric distribution

$X \sim g(p)$,
PMF of X:

$$p_X(k) = (1-p)^{k-1}p, \quad k = 1, 2, 3, \dots$$

$$\mathbb{E}(X) = \sum_{k=1}^{\infty} k p_X(k) = \sum_{k=1}^{\infty} k(1-p)^{k-1}p = \sum_{k=1}^{\infty}\sum_{j=1}^{k}(1-p)^{k-1}p =$$

$$p\sum_{j=1}^{\infty}\sum_{k=j}^{\infty}(1-p)^{k-1} = p\sum_{j=1}^{\infty}\frac{(1-p)^{j-1}}{1-(1-p)} = \sum_{j=1}^{\infty}(1-p)^{j-1} = \frac{1}{p}.$$

Example

Flip a fair coin until you get heads for the first time. What is the expected number of tosses?

Solution

$\mathbb{E}(X) = \frac{1}{\frac{1}{2}} = 2.$

$\mathbb{E}(X)$ of Poisson distribution

$X \sim \mathcal{P}(\lambda)$,
PMF of X:

$$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \dots,$$

$$\begin{aligned}\mathbb{E}(X) = \sum_{k=0}^{\infty} k e^{-\lambda}\frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} e^{\lambda}\frac{\lambda^k}{(k-1)!} &= e^{-\lambda}\lambda\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!} \\ &= e^{-\lambda}\lambda\sum_{j=0}^{\infty}\frac{\lambda^j}{j!} = e^{-\lambda}\lambda e^{\lambda} = \lambda.\end{aligned}$$

Expected values of functions of random variables

Let X be a discrete random variable with PMF p_X and support S_X.

Theorem (Law of the unconscious statistician - LOTUS)

Let $g(X)$ be a real valued function of X. Then, the expected value of the random variable $g(X)$ is given by

$$\mathbb{E}(g(X)) = \sum_{x \in S_X} g(x) p_X(x).$$

This is well-defined as long as $\sum_{x \in S_X} |g(x)| p_X(x)$ converges.

Example

Roll a four sided die. Let X be the number of spots on the die. Suppose the payoff function is given by $Y = X^2 - 3X + 1$. Is this a good bet?

Solution

Since the expected payoff is positive, the bet looks like worth taking.

$$\mathbb{E}(Y) = \mathbb{E}(X^2 - 3X + 1) = \mathbb{E}(X^2) - 3\mathbb{E}(X) + 1,$$

$$\mathbb{E}(X) = \frac{1}{4}(1 + 2 + 3 + 4) = \frac{5}{2},$$

$$\mathbb{E}(X^2) = \frac{1}{4}(1^2 + 2^2 + 3^2 + 4^2) = \frac{15}{2},$$

so $\mathbb{E}(Y) = 1$.

Remark (Moments of random variable X)

- ***second moment*** *of* X: $\mathbb{E}(X^2)$,
- ***kth moment*** *of* X: $\mathbb{E}(X^k)$ - *the expected value of the random variable* X^k.

Variance and standard deviation

Definition

If X is a random variable with mean $\mathbb{E}(X)$, then

- the **variance** of X is defined by

$$Var(X) = \mathbb{E}\left(X - \mathbb{E}(X)\right)^2,$$

- the **standard deviation** of X is defined by

$$\sigma_X = \sqrt{Var(X)}.$$

If X takes values in S_X with probability mass function (PMF) p_X, then

$$Var(X) = \sum_{x \in S_X} (x - \mathbb{E}(X))^2 p_X(x)$$

Remark

- *The variance is a weighted average of the squared distance to the mean.*
- *By definition, we are weighting high probability values more then low probability values*
- *The variance is always nonnegative.*

The **variance** and the **standard deviation** provides a measure of dispersion of X around its mean.

- The standard deviation σ_X has the same units as X.
- The variance has the same units as X^2 (X in meters $\implies$ $Var(X)$ in meters squared).

Therefore the standard deviation is a natural measure of spread.

Properties of variance

Theorem

1. $$\forall a, b \in \mathbb{R} : \quad Var(aX + b) = a^2 Var(X).$$

2. $$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2.$$

Example

Compute variances of the following random variables:

- X, $S_X = \{1, 2, 3, 4, 5\}$:

$$p_X(1) = p_X(2) = p_X(3) = p_X(4) = p_X(5) = \frac{1}{5},$$

- Y, $S_Y = \{1, 2, 3, 4, 5\}$:

$$p_Y(1) = p_Y(5) = \frac{1}{10},\ p_Y(2) = p_Y(4) = \frac{2}{10},\ p_Y(3) = \frac{4}{10},$$

- Z, $S_Z = \{1, 5\}$:

$$p_Z(1) = p_Z(5) = \frac{1}{2},$$

- W, $S_W = \{3\}$:

$$p_W(3) = 1.$$

Solution

$$\mathbb{E}X = \mathbb{E}Y = \mathbb{E}Z = \mathbb{E}W = 3,$$

$$Var(X) = \frac{1}{5}\sum_{k=1}^{5}(k-3)^2 = 2,$$

$$Var(Y) = 2 \cdot \frac{1}{10} \cdot 2^2 + 2 \cdot \frac{1}{5} \cdot 1^2 = 1.2,$$

$$Var(Z) = 2 \cdot \frac{1}{2} 2^2 = 4,$$

$$Var(W) = 0.$$