Lecture 7

Continuous case

Definition

The **expected value or mean** of a continuous random variable X is defined by

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx,$$

if $\int_{-\infty}^{\infty} |x| f_X(x)\,dx < \infty$.

$\sum_{x \in S_X} x\, p_X(x)$ discrete

Remark (How do we interpret the expected value?)

The expected value of the r.v. X is the center of gravity of the mass distribution described by the function f_X.

Example

(1) Let $X \sim \mathcal{U}[a,b]$. Find $\mathbb{E}(X)$.

Solution

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_a^b x \frac{1}{b-a} dx = \frac{a+b}{2}.$$

- *the mean is at the midpoint of the range.*

(2) Let X has a density $f_X(x) = \frac{3}{8}x^2 \mathbb{I}_{(0,2)}(x)$. Find $\mathbb{E}(X)$.

Solution

$$\mathbb{E}(X) = \int_0^2 x \frac{3}{8} x^2 dx = \frac{3}{2}.$$

Example (cont'd)

(3) Let $X \sim Exp(\lambda)$. Find $\mathbb{E}(X)$.

Solution

$S_X = [0, \infty)$ and its density is $f(x) = \lambda e^{-\lambda x} \mathbb{I}_{(0,\infty)}(x)$
$\implies \mathbb{E}(X) = \frac{1}{\lambda}$.

(4) Let $Z \sim \mathcal{N}(0, 1)$. Find $\mathbb{E}(Z)$.

Solution

$S_Z = \mathbb{R}$ and its density function is

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}.$$

Note that the pdf is symmetric around 0, so its mean must be 0 (if it exists) $\implies \mathbb{E}(Z) = 0$.

Remark (The expected value may not exist!)

Let X *be a continuous random variable with pdf*

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad x \in \mathbb{R}.$$

The pdf is symmetrical around 0 *but the expected value does not exist:*

$$\int_{-\infty}^{\infty} |x| f_X(x) dx = +\infty.$$

∫

Properties of $\mathbb{E}(X)$

Analogous properties as for the discrete case:

1. If X and Y are random variables on a sample space Ω then

$$\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y). \tag{1}$$

2. If a and b are constants then

$$\mathbb{E}(aX+b) = a\mathbb{E}(X) + b. \tag{2}$$

Expectation of functions of X

Theorem (LOTUS)

Let X be a continuous random variable with probability density function f_X and let $g : \mathbb{R} \to \mathbb{R}$ be a function, then

$$\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x) f(x) dx,$$

if $\int_{-\infty}^{\infty} |g(x)| f_X(x) dx < \infty$.

Example

Let $X \sim Exp(\lambda)$. Find $\mathbb{E}(X^2)$.

Solution

$$\mathbb{E}(X^2) = \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx = \ldots = \frac{2}{\lambda^2}.$$

Example

Let $X \sim U[0,1]$. The value of X and the point $\frac{1}{2}$ split the interval $[0,1]$ into three parts. What are the expected lengths of these three intervals?

Solution

$Y = \min(X, \frac{1}{2})$, $U = |X - \frac{1}{2}|$, $W = \min(\frac{1}{2}, 1 - X)$
$\mathbb{E}(Y) =?$, $\mathbb{E}(U) =?$, $\mathbb{E}(W) =?$

$$\mathbb{E}(Y) = \mathbb{E}(\min\left(X, \frac{1}{2}\right)) = \int_0^{\frac{1}{2}} x dx + \int_{\frac{1}{2}}^1 \frac{1}{2} dx = \frac{3}{8},$$

$$\mathbb{E}(U) = \mathbb{E}|X - \frac{1}{2}| = \int_0^{\frac{1}{2}} (\frac{1}{2} - x) dx + \int_{\frac{1}{2}}^1 (x - \frac{1}{2}) dx = \frac{1}{4},$$

$$\mathbb{E}(W) = \mathbb{E} \min\left(\frac{1}{2}, 1 - X\right) = \int_0^{\frac{1}{2}} \frac{1}{2} dx + \int_{\frac{1}{2}}^1 (1 - x) dx = \frac{3}{8}.$$

Variance $Var(X)$

Definition

The variance of a random variable X with density f_X is defined by

$$Var(X) = \mathbb{E}\left(X - \mathbb{E}(X)\right)^2 = \int_{-\infty}^{\infty} (x - \mathbb{E}(X))^2 f(x)dx.$$

and standard deviation $\sigma_X = \sqrt{Var(X)}$.

Properties of variance:

- For constants a and b:

$$Var(aX + b) = a^2 Var(X),$$

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$$Var(X) = \mathbb{E}(X^2) - \left(\mathbb{E}(X)\right)^2 .$$

Example

1. Let $X \sim \mathcal{U}[a, b]$. Find $Var(X)$.

Solution

$\mathbb{E}(X) = \frac{a+b}{2}$, $\mathbb{E}(X^2) = \int_a^b x^2 \frac{1}{b-a} dx = \frac{a^2+ab+b^2}{3}$

$$\implies Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{(b-a)^2}{12}.$$

2. Let $X \sim Exp(\lambda)$. Find $Var(X)$ and σ_X.

Solution

$\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$, $\sigma_X = \frac{1}{\lambda}$.

$\mathbb{E}Z = 0$, $Var Z = 1$

Example

Let $Z \sim \mathcal{N}(0,1)$. Prove that $Var(Z) = 1$.

Solution

Since $\mathbb{E}(Z) = 0$,

$$Var(Z) = \mathbb{E}(Z^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2}} dz = 1.$$

Functions of random variables

Let X be a random variable. We want to start looking at $Y = g(X) = g \circ X$, where $g : \mathbb{R} \to \mathbb{R}$.

Remark

If $Y = g(X)$ is a function of a random variable X then Y is also a random variable, since it provides a numerical value for each possible outcome.

The question is: how can we find **the distribution of** Y knowing **the distribution of** X?

Theorem

Let X be a discrete random variable. Then $Y = g(X)$ is also a discrete random variable and

- $S_Y = g(S_X)$ *and*
- $$p_Y(y) = \sum_{x \in S_X : g(x) = y} p_X(x).$$

Example

Let X be a discrete random variable with the pmf:
$p_X(-1) = \frac{1}{5}$, $p_X(0) = \frac{2}{5}$ *and* $p_X(1) = \frac{2}{5}$.
Let $g(x) = x^2$. Find the distribution of $Y = g(X)$.

Solution

$S_X = \{-1, 0, 1\} \implies S_Y = \{0, 1\}$.

$$p_Y(1) = \mathbb{P}(Y = 1) = \mathbb{P}(X^2 = 1) = p_X(-1) + p_X(1) = \frac{3}{5}.$$

$$p_Y(0) = p_X(0) = \frac{2}{5}.$$

Example

Let $X \sim g(p)$ and $g(x) = \left\lfloor \frac{x}{2} \right\rfloor$. Find the distribution of Y.

Solution

$$S_X = \mathbb{N} \implies S_Y = \mathbb{N} \cup \{0\},$$

$$p_Y(0) = \mathbb{P}(Y = 0) = \mathbb{P}(X = 1) = p,$$

$$p_Y(k) = \mathbb{P}(X = 2k) + \mathbb{P}(X = 2k + 1) = p(2 - p)(1 - p)^{2k-1}, \quad k \in \mathbb{N}.$$

Check whether p_Y is a well defined probability mass function?

Some functions of a continuous random variable turn out to be discrete random variables.

Example

$X \sim \mathcal{U}[-10, 10]$; $g(x) = \text{sign}(x)$. Then

$$p_Y(-1) = \mathbb{P}(X < 0) = \frac{1}{2},$$

$$p_Y(0) = \mathbb{P}(X = 0) = 0,$$

$$p_Y(1) = \mathbb{P}(X > 0) = \frac{1}{2}.$$

Hence Y has two point distribution ($S_Y = \{-1, 1\}$).

Remark

If X is a continous random variable, then Y is not necessarily a continuous one!

In each of the following examples we will start from finding a cumulative distribution function of the random variable $Y = g(X)$ (cdf - technique).

Example

Let $X \sim \mathcal{U}[0,1]$ and $Y = X^2$. Find F_Y.

Solution

$S_X = [0,1] \implies S_Y = [0,1]$.

For $0 \le y < 1$*:*

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(X^2 \le y) = \mathbb{P}(X \le \sqrt{y}) = \int_0^{\sqrt{y}} f_X(x)dx$$

$$= \int_0^{\sqrt{y}} 1 dx = \sqrt{y}.$$

Therefore

$$F_Y(y) = \begin{cases} 0, \ y < 0, \\ \sqrt{y}, \ 0 \le y < 1, \\ 1, \ y \ge 1. \end{cases}$$

Example

Let $X \sim Exp(1)$ and $Y = \sqrt{X}$. Find f_Y.

Solution

$S_X = [0, \infty) \implies S_Y = [0, \infty)$. *For* $y > 0$*:*

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(\sqrt{X} \leq y) = \mathbb{P}(X \leq y^2) = \int_0^{y^2} f_X(x)dx$$
$$= \int_0^{y^2} e^{-x}dx = 1 - e^{-y^2}.$$

Hence

$$F_Y(y) = \begin{cases} 0, & y < 0, \\ 1 - e^{-y^2}, y \geq 0. \end{cases}$$

By taking a derivative of F*, we get*

$$f_Y(y) = \begin{cases} 0, & y < 0, \\ 2ye^{-y^2}, & y \geq 0. \end{cases}$$

Example

Let X be a random variable with probability density function:
$f_X(x) = \frac{x^2}{3}\mathbb{I}_{(-1,2)}(x)$ and let $Y = |X|$. Find f_Y.

Solution

$S_Y = [0, 2]$.

$$F_Y(t) = \begin{cases} 0, & t \leq 0, \\ \int_{-t}^{t} \frac{x^2}{3} dx = \frac{2t^3}{9}, & 0 \leq t < 1, \\ \int_{-1}^{t} \frac{x^2}{3} dx = \frac{t^3}{9} + 1, & 1 \leq t < 2, \\ 1, & t \geq 2. \end{cases}$$

$f_Y(y) = F'_Y(y)$:

$$f_Y(t) = \begin{cases} \frac{2}{3}t^2, & t \in (0,1), \\ \frac{t^2}{3}, & t \in [1,2), \\ 0, & t \notin (0,2). \end{cases}$$

Standardizing a normal random variable

Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Find the distributon of $Y = \frac{X-\mu}{\sigma}$.

Solution

$$\mathbb{P}(Y \leq t) = \mathbb{P}(\frac{X-\mu}{\sigma} \leq t) = \mathbb{P}(X \leq \mu + t\sigma) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\mu+t\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

substituting $y = \frac{x-\mu}{\sigma}$, *we get*

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left(-\frac{y^2}{2}\right) dy = \Phi(t),$$

where Φ denotes the cdf of the standard normal distribution, $\mathcal{N}(0,1)$.

Example

Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Prove that $\mathbb{E}(X) = \mu$ and $Var(X) = \sigma^2$.

Solution

- *From Example(4) we know that if $Z \sim \mathcal{N}(0,1)$, then $\mathbb{E}(Z) = 0$.*
- *For $X \sim \mathcal{N}(\mu, \sigma^2)$ the random variable $Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$.*

Therefore $X = \sigma Z + \mu$ and by linearity we have

$$\mathbb{E}(X) = \mathbb{E}(\sigma Z + \mu) = \sigma \mathbb{E} Z + \mu = \mu,$$

and

$$Var(X) = Var(\sigma Z + \mu) = \sigma^2 Var(Z) = \sigma^2.$$

$\mathrm{Var}\, Y = \mathrm{Var}(aX+b) = a^2 \mathrm{Var}\, X = a^2\sigma^2$

Remark

A linear function of a normal random variable is normal.
If $X \sim \mathcal{N}(\mu, \sigma^2)$ *and* $Y = aX + b$, *then* $Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.

$\mathbb{E}Y$ $\quad$ $\mathrm{Var}(Y)$

$Y = aX + b$

$\mathbb{E}Y = a\mathbb{E}X + b = a\mu + b$