# Lecture 8

# Joint distributions

**Discrete case**

## Definition

Let $X$ and $Y$ be discrete random variables defined on the same probability space $\Omega$. The function:

$$p_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$$

is called the **joint probability mass function** of $X$ and $Y$.

## Example

Fair coin is tossed 3 times. Let

- $X$ = number of heads in the first 2 tosses,
- $Y$ = number of heads in all 3 tosses.

We can display the joint pmf in a table:

| X / Y | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 1/8 | 1/8 | 0 | 0 |
| 1 | 0 | 1/4 | 1/4 | 0 |
| 2 | 0 | 0 | 1/8 | 1/8 |

A **joint probability mass function** must satisfy two properties:

1. $0 \leq p_{X,Y}(x,y) \leq 1$
2. The total probability is 1. We can express this as a double sum:

$$\sum_{x \in S_X} \sum_{y \in S_Y} p_{X,Y}(x,y) = 1.$$

## Example

Joint probability mass function:

$$p(x, y) = \begin{cases} c(x+y), & x = 1, 2,\ y = 1, 2, \\ 0, & otherwise. \end{cases}$$

Determine the value of $c$.

## Solution

$$p(1,1) + p(1,2) + p(2,1) + p(2,2) = 1,$$

*hence*

$$c\,(1+1+1+2+2+1+2+2) = 12c = 1 \implies c = \frac{1}{12}.$$

## Remark

*The joint PMF determines the probability of any event that can be specified in terms of the random variables $X$ and $Y$:*

$$\mathbb{P}((X,Y)\in A) = \sum_{(x,y)\in A} p_{X,Y}(x,y).$$

## Example

Roll two symmetric dice. Let $X$ be the outcome on the first die and $Y$ on the second die. Let $B = \{(x,y) : y - x \geq 2\}$. Find $\mathbb{P}((X,Y)\in B)$.

## Solution

*Both $X$ and $Y$ take values in $\{1,2,3,4,5,6\}$ and the joint pmf is*
$p(i,j) = \frac{1}{36}$, $i,j = 1,\ldots,6$,
$B = \{(1,3),(1,4),(1,5),(1,6),(2,4),(2,5),(2,6),(3,5),(3,6),(4,6)\}$
$\implies \mathbb{P}((X,Y)\in B) = \frac{10}{36}$.

# Marginal distributions

From the joint PMF $p_{X,Y}$ we can calculate the **marginal pmf's** of $X$ and $Y$:

$$p_X(x) = \sum_y p_{X,Y}(x,y) \quad \textit{and} \quad p_Y(y) = \sum_x p_{X,Y}(x,y).$$

## Example

The joint PMF of $X$ and $Y$:

| X \ Y | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 1 | 1/20 | 2/20 | 2/20 | |
| 2 | 2/20 | 4/20 | 1/20 | 2/20 |
| 3 | | 1/20 | 3/20 | 1/20 |
| 4 | | 1/20 | | |

- $p_X(3) = ?$.
- Find marginal pmf's of $X$ and $Y$.

## Solution

*Marginal PMF of X:*

$$p_X(1) = \sum_y p(1,y) = p(1,0) + p(1,1) + p(1,2) + p(1,3) = \frac{1}{20} + \frac{2}{20} + \frac{2}{20} = \frac{1}{4}$$

$$p_X(2) = p(2,0) + p(2,1) + p(2,2) + p(2,3) = \frac{2}{20} + \frac{4}{20} + \frac{1}{20} + \frac{2}{20} == \frac{9}{20},$$

$$p_X(3) = p(3,0) + p(3,1) + p(3,2) + p(3,3) = \frac{1}{20} + \frac{3}{20} + \frac{1}{20} = \frac{1}{4},$$

$$p_X(4) = \frac{1}{20}.$$

*Marginal PMF of Y:*

$$p_Y(0) = p(1,0) + p(2,0) = \frac{1}{20} + \frac{2}{20} = \frac{3}{20},$$

$$p_Y(1) = \frac{2}{20} + \frac{4}{20} + \frac{1}{20} + \frac{1}{20} = \frac{8}{20},$$

$$p_Y(2) = \frac{2}{20} + \frac{1}{20} + \frac{3}{20} = \frac{6}{20}, \quad p_Y(3) = \frac{2}{20} + \frac{1}{20} = \frac{3}{20}.$$

## Remark

*Marginal probability mass functions don't determine the joint pmf.*

## Example

| X \ Y | 0 | 1 |
|---|---|---|
| 0 | 0 | 1/2 |
| 1 | 1/2 | 0 |

| $X \backslash Y$ | 0 | 1 |
|---|---|---|
| 0 | 1/4 | 1/4 |
| 1 | 1/4 | 1/4 |

Two different joint pmfs have the same marginal distributions of $X$ and $Y$.

# Independence of random variables

**Definition**

Two random variables $X$ and $Y$ defined on the same probability space $(\Omega, \mathbb{P})$ are **independent** if

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B), \quad \forall A, B \subset \mathbb{R}.$$

# Discrete case

Two discrete random variables $X$ and $Y$ (defined on the same space $(\Omega, \mathbb{P})$) are **independent**, if

$$\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x)\mathbb{P}(Y = y) \ \ \forall x, y \in \mathbb{R},$$

or the same condition in terms of pmf:

$$p_{X,Y}(x, y) = p_X(x)p_Y(y), \ \ \forall x, y \in \mathbb{R}.$$

## Example

Joint PMF of $(X, Y)$:

| $X \backslash Y$ | 0 | 1 |
|---|---|---|
| 0 | 2/6 | 1/6 |
| 1 | 2/6 | 1/6 |

. Are $X$ and $Y$ independent?

## Solution

$$p_X(0) = \frac{3}{6}, \quad p_X(1) = \frac{3}{6}, p_Y(0) = \frac{4}{6}, \quad p_Y(1) = \frac{2}{6},$$

*so*

$$p_{X,Y}(0,0) = p_X(0)p_Y(0),$$

$$p_{X,Y}(0,1) = p_X(0)p_Y(1),$$

$$p_{X,Y}(1,0) = p_X(1)p_Y(0),$$

$$p_{X,Y}(1,1) = p_X(1)p_Y(1),$$

*hence $X$ and $Y$ are independent.*

## Example

Joint PMF of (X,Y):

| $X\backslash Y$ | 0 | 1 |
| --- | --- | --- |
| 0 | 1/2 | 0 |
| 1 | 1/4 | 1/4 |

. Are $X$ and $Y$ independent?

## Solution

$$p_X(0) = \frac{1}{2}, \quad p_X(1) = \frac{1}{2}, \quad p_Y(0) = \frac{3}{4}, \quad p_Y(1) = \frac{1}{4},$$

$$p_{X,Y}(0,1) = 0 \neq p_X(0)p_Y(1) = \frac{1}{8},$$

*hence $X$ and $Y$ are not independent.*