Lecture 9

Continuous case

Definition

Let X and Y be continuous random variables. Then $f_{X,Y}$ is the **joint density function** of X and Y, if for any region $A \subset \mathbb{R}^2$

$$\mathbb{P}((X,Y) \in A) = \int\int_A f_{X,Y}(x,y)dxdy,$$

in particular, if $A = \{(x,y) : a \leq x \leq b, c \leq y \leq d\}$, then

$$\mathbb{P}((X,Y) \in A) = \mathbb{P}(a \leq X \leq b, c \leq Y \leq d)$$
$$= \int_a^b \int_c^d f_{X,Y}(x,y)dydx$$

Remark

- *A joint probability denstiy function must satisfy:*
  1. $f(x, y) \geq 0$, *for all* x, y,
  2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dxdy = 1$.
- *The probability that* (X, Y) *lies on any interval, any curve, sraight line or point is* 0.
- $$\mathbb{P}(X = a, Y = b) = 0$$

  and

  $$\mathbb{P}(a \leq X \leq b, c \leq Y \leq d) = \mathbb{P}(a < X \leq b, c < Y \leq d)$$
  $$= \mathbb{P}(a \leq X \leq b, c < X < d) = \mathbb{P}(a < X < b, c < X < d).$$

Definition (uniform distribution on D)

A random vector (X, Y) has the uniform ditribution on $D \subset \mathbb{R}^2$ if

$$f(x,y) = \begin{cases} \frac{1}{|D|}, & (x,y) \in D, \\ 0, & \text{otherwise}, \end{cases}$$

where $|D|$ denotes the area of D.

Example

Let $f_{X,Y}$ be the joint probability density function of X and Y:

$$f(x,y) = \begin{cases} 1, & x,y \in [0,1], \\ 0, & \text{otherwise}. \end{cases}$$

Find $\mathbb{P}(X > Y)$.

Solution

$$\mathbb{P}(X > Y) = \int\int_{\{(x,y):x>y\}} f(x,y)dxdy = \int_0^1 \int_y^1 1dxdy = \int_0^1 (1-y)dy = \frac{1}{2}.$$

Example

Let $f_{X,Y}$ be the joint probability density function of X and Y:

$$f(x,y) = \begin{cases} 4xy, & x,y \in [0,1], \\ 0, & \text{otherwise}. \end{cases}$$

Find $\mathbb{P}(X < 0.5, Y > 0.5)$.

Solution

$$\mathbb{P}(X < 0.5, Y > 0.5) = \int_0^{0.5} \int_{0.5}^{1} 4xydydx = \frac{3}{2} \int_0^{0.5} xdx = \frac{3}{16}.$$

Marginal distributions

Definition

The marginal probability density function of continuous random variables X and Y are

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy \text{ and } f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

Example

Let (X, Y) be drawn uniformly from the triangle:

$$T = \{(x, y) : 0 \leq x + y \leq 3, x \geq 0, y \geq 0\}$$

- Find the joint pdf of X and Y.
- Find the marginal pdfs.

Solution

- $$f(x, y) = \begin{cases} \frac{2}{9}, & (x, y) \in T, \\ 0, & (x, y) \notin T. \end{cases}$$
- $$f_X(x) = \begin{cases} \int_0^{3-x} \frac{2}{9} dy = \frac{2}{9}(3 - x), & x \in (0, 3), \\ 0, & x \notin (0, 3). \end{cases}$$
- $$f_Y(y) = \begin{cases} \frac{2}{9}(3 - y), & y \in (0, 3), \\ 0, & y \notin (0, 3). \end{cases}$$

Example

Let $f_{X,Y}$ be the joint probability density function of random variables X and Y:

$$f(x,y) = \begin{cases} cxy, & 0 \leq x \leq y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

- Find c.
- Determine the marginal densities for X and Y.

Solution

- $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1 \implies \int_0^1\int_x^1 cxydydx = 1 \implies c = 8.$$
- $$f_X(x) = \begin{cases} \int_x^1 8xydy = 4x(1-x^2), & x \in (0,1), \\ 0, & x \notin (0,1), \end{cases} \quad f_Y(y) = \begin{cases} \int_0^y 8xydx = 4y^3, & y \in (0,1), \\ 0, & y \notin (0,1). \end{cases}$$

Cumulative distribution function - continuous case

The joint cumulative distribution function of X and Y is defined as

$$F_{X,Y}(x,y) = \mathbb{P}(X \leq x, Y \leq y).$$

If X and Y are described by a joint pdf $f_{X,Y}$, then

$$F_{X,Y}(x,y) = \mathbb{P}(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v) du dv.$$

Conversely,

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}}{\partial x \partial y}(x,y).$$

Example

We are told that the joint pdf of the random variables X and Y is constant c on the square $[0,1]^2$ (and 0 outside the square). Find the value of c and the joint cdf of X and Y.

Solution

- $c = 1$,
-

$$F(s,t) = \begin{cases} 0, & s < 0 \ \ or \ \ t < 0, \\ st, & 0 \le s < 1, 0 \le t < 1, \\ s, & 0 \le s < 1, t \ge 1, \\ t, & s \ge 1, 0 \le t < 1, \\ 1, & s \ge 1, t \ge 1. \end{cases}$$

The marginal distribution via CDF

Another way to derive the marginal distributions from the joint is to use the cumulative distribution function:

$$F_X(x) = \lim_{y \to \infty} F_{X,Y}(x, y) \quad \text{i} \quad F_Y(y) = \lim_{x \to \infty} F_{X,Y}(x, y).$$

Indepedence of random variables

Definition

Random variables X and Y defined on the same probability space $(\Omega, \mathbb{P})$ are **independent**, if:

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B), \quad \forall A, B \subset \mathbb{R}.$$

In the framework of the CDFs:

Theorem

A random vector (X, Y) has independent components, i.e. X and Y are independent, if

$$F_{X,Y}(x, y) = F_X(x)F_Y(y), \quad \forall x, y \in \mathbb{R}.$$

Continuous case

Two continuous random variables X and Y are **independent** if their joint probability density function is the product of the marginal pdfs.

$$f_{X,Y}(x,y) = f_X(x)f_Y(y), \ \ \forall x, y \in \mathbb{R}$$

Example

The joint *pdf* of (X, Y) is given by

$$f_{X,Y}(x,y) = \begin{cases} 12xy(1-y), & x, y \in [0,1], \\ 0, & otherwise. \end{cases}$$

Are X and Y independent?

Solution

$$f_{X,Y}(x,y) = 12xy(1-y)\mathbb{I}_{[0,1]\times[0,1]}(x,y)$$
$$= 2x\mathbb{I}_{(0,1)}(x) \cdot 6y\mathbb{I}_{(0,1)}(y) = f_X(x)f_Y(y),$$

$\Longrightarrow$ *X and Y are independent.*

Example

The joint pdf of (X, Y) is given by:

$$f_{X,Y}(x,y) = \begin{cases} x + \frac{1}{4}y, & , x \in [0,1], y \in [0,2], \\ 0, & \text{otherwise} \end{cases}$$

Are X and Y independent?

Solution

$$f_X(x) = \begin{cases} \int_0^2 (x + \frac{1}{4}y)dy = 2x + \frac{1}{2}, & x \in (0,1), \\ 0, & x \notin (0,1), \end{cases}$$

$$f_Y(y) = \begin{cases} \int_0^1 (x + \frac{1}{4}y)dx = \frac{1}{4}y + \frac{1}{2}, & y \in (0,2), \\ 0, & y \notin (0,2), \end{cases}$$

hence $f_{X,Y}(x,y) \neq f_X(x)f_Y(y) \implies X$ *and* Y *are not independent.*

Example

The joint *pdf* of (X, Y) is given by

$$f_{X,Y}(x, y) = \begin{cases} 2, & 0 \leq x \leq y \leq 1, \\ 0, & otherwise. \end{cases}$$

Are X and Y independent?

Solution

The support of (X, Y)*:* $\{(x, y) : 0 \leq x \leq y \leq 1\}$ *can not be written as the product of two sets , i.e.*

$$S_{X,Y} \neq S_X \times S_Y.$$

Hence

$$f_{X,Y}(x, y) \neq f_X(x)f_Y(y), \quad \forall x, y \in \mathbb{R}$$

the LHS and the RHS are nonzero on different regions on the R^2 *plane* $\Longrightarrow$ X *and* Y *are not independent.*

Example

A random vector (X, Y) is uniformly distributed on the circle (disc): $\{(x, y) : x^2 + y^2 \leq 1\}$. Are X and Y independent?

Solution

Note that the support of (X, Y):

$$S_{X,Y} = \{(x, y) : x^2 + y^2 \leq 1\},$$

can not be written as a product of the supports of X and Y:

$$S_{X,Y} = \{(x, y) : x^2 + y^2 \leq 1\} \neq S_X \times S_Y = [-1, 1] \times [-1, 1],$$

$\implies$ *X i Y are not independent.*

Example (Competing exponentials)

Let $X \sim Exp(\lambda_1)$ and $Y \sim Exp(\lambda_2)$ be independent. Find $\mathbb{P}(X < Y)$.

Solution

$$\mathbb{P}(X < Y) = \int\int_{\{(x,y):x>0,y>0 \text{ and } x<y\}} \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y} dxdy$$

$$= \int_0^\infty \left(\int_0^y \lambda_1 e^{-\lambda_1 x} \right) \lambda_2 e^{-\lambda_2 y} dy = \int_0^\infty (1 - e^{-\lambda_1 y}) \lambda_2 e^{-\lambda_2 y} dy$$

$$= 1 - \int_0^\infty \lambda_2 e^{-(\lambda_1+\lambda_2)y} dy = 1 - \frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$