Lecture 10

Functions of bivariate random vectors

Let (X, Y) be a bivariate random vector and $g : \mathbb{R}^2 \to \mathbb{R}$. Knowing the distribution of (X, Y) we can determine the distribution of a new random variable $Z = g(X, Y)$.

Discrete case

Let (X, Y) be a discrete random vector with the joint pmf $p_{X,Y}$ and $Z = g(X, Y)$, then

$$p_Z(z) = \sum_{\{(x,y):g(x,y)=z\}} p_{X,Y}(x, y)$$

Example

The pmf of (X, Y) is given by:

X \ Y	0	2
-1	1/6	1/3
1	1/6	1/3

. Find the pmf of $Z = \cos\left(\frac{\pi}{3}(X+Y)\right)$.

Solution

$$S_Z = \left\{\cos\left(-\frac{\pi}{3}\right), \cos\left(\frac{\pi}{3}\right), \cos(\pi)\right\} = \left\{\frac{1}{2}, -1\right\}$$

$$\mathbb{P}\left(Z = \frac{1}{2}\right) = \mathbb{P}(X+Y=-1) + \mathbb{P}(X+Y=1) = \frac{1}{6} + \frac{1}{3} + \frac{1}{6} = \frac{2}{3},$$

$$\mathbb{P}(Z=-1) = \mathbb{P}(X+Y=3) = \frac{1}{3}.$$

Convolution - discrete case

Let $Z = X + Y$, where X and Y are independent random variables with pmfs: p_X and p_Y. Then:

$$p_Z(z) = \mathbb{P}(X + Y = z) = \sum_{\{(x,y):x+y=z\}} \mathbb{P}(X = x, Y = y)$$
$$= \sum_x \mathbb{P}(X = x, Y = z - x) = \sum_x p_X(x) p_Y(z - x),$$

the resulting pmf p_Z is called the **convolution** of p_X and p_Y.

Example

Let X, Y be independent random variables $X \sim \mathcal{P}(\lambda_1)$ and $Y \sim \mathcal{P}(\lambda_2)$. Define $Z = X + Y$. Show that $Z \sim \mathcal{P}(\lambda_1 + \lambda_2)$.

Solution

$$\mathbb{P}(Z=0) = \mathbb{P}(X+Y=0) = \mathbb{P}(X=0, Y=0) = \mathbb{P}(X=0)\mathbb{P}(Y=0) \\ = e^{-\lambda_1 - \lambda_2},$$

$$\mathbb{P}(Z=k) = \mathbb{P}(X+Y=k) = \sum_{j=0}^{k} \mathbb{P}(X=j, Y=k-j)$$

$$= \sum_{j=0}^{k} \mathbb{P}(X=j)\mathbb{P}(X=k-j) = \sum_{j=0}^{k} e^{-\lambda_1} \frac{\lambda_1^j}{j!} e^{-\lambda_2} \frac{\lambda_2^{k-j}}{(k-j)!}$$

$$= e^{-\lambda_1 - \lambda_2} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} \lambda_1^j \lambda_2^{k-j} = e^{-\lambda_1 - \lambda_2} \frac{(\lambda_1 + \lambda_2)^k}{k!}, \quad k = 1, 2, 3, \ldots.$$

Mixed case

Example

Let X, Y be independent random variables such that $X \sim \mathcal{U}[0,1]$ and $Y \sim B\left(\frac{1}{2}\right)$. Find the cumulative distribution function of $Z = X + Y$.

Solution

$S_Z \subset [0,2]$ *and*

$$F_Z(t) = \begin{cases} 0, & t < 0, \\ \frac{1}{2}t, & 0 \leq t < 2, \\ 1, & t \geq 2, \end{cases}$$

$\implies Z \sim \mathcal{U}[0,2]$.

Remark

If X and Y are independent random variables such that X is continuous and Y is discrete then $X + Y$ has continuous distribution.

Continuous case

Let (X, Y) be a bivariate random vector with a joint pdf $f_{X,Y}$. Consider a function $g : \mathbb{R}^2 \to \mathbb{R}$ and a random variable $Z = g(X, Y)$. Then the cdf of Z is of the form:

$$F_Z(z) = \mathbb{P}(g(X,Y) \le z) = \iint\limits_{\{(x,y):g(x,y)\le z\}} f_{X,Y}(x,y)dxdy.$$

Example

Let $(X, Y) \sim \mathcal{U}(D)$, where $D = \{(x,y) \in \mathbb{R}^2 : x \in [0,1] \text{ and } 0 \le y \le x\}$. Find pdf of $Z = Y - X$.

Solution

$S_Z = [-1, 0]$ *and*

$$F_Z(z) = \begin{cases} 0, & z < -1, \\ (1+z)^2, & -1 \le z < 0, \\ 1, & z \ge 0, \end{cases} \implies f_Z(z) = \begin{cases} 2(1+z), & z \in [-1,0], \\ 0, & \mathit{otherwise}. \end{cases}$$

Example

Let X, Y be intependent identically distributed (i.i.d.) random variables $X, Y \sim Exp(1)$ and let $Z = \min(X, Y)$. Determine the pdf of Z.

Solution

$$F_Z(z) = \begin{cases} 0, & z < 0, \\ 1 - e^{-2z}, & z \geq 0, \end{cases} \implies f_Z(z) = \begin{cases} 2e^{-2z}, & z \geq 0, \\ 0, & otherwise. \end{cases}$$

Example

Let X, Y be i.i.d. random variables $X, Y \sim \mathcal{U}[0,1]$. Determine the pdf of $Z = XY$.

Solution

$$S_Z = [0,1], \quad F_Z(z) = 1 - \mathbb{P}(Z > z) = 1 - \mathbb{P}(XY > z) = 1 - \int_z^1 \int_{\frac{z}{x}}^1 dydx$$

$$\implies \quad f_z(z) = -\ln(z)\mathbb{I}_{[0,1]}(z).$$

Convolution - continuous case

Let X and Y be independent continuous random variables with pdfs f_X and f_Y, respectively. We will find the pdf of $W = X + Y$, by first finding its cdf and then differentiating:

$$F_W(w) = \mathbb{P}(X + Y \leq w) = \int_{-\infty}^{\infty} \int_{-\infty}^{w-x} f_X(x) f_Y(y) dy dx$$
$$= \int_{-\infty}^{\infty} f_X(x) F_Y(w - x) dx.$$

Then pdf of W is obtained by differentiating the cdf:

$$f_W(w) = \frac{dF_W}{dw}(w) = \int_{-\infty}^{\infty} f_X(x) \frac{d}{dw} F_Y(w - x) dx$$
$$= \int_{-\infty}^{\infty} f_X(x) f_Y(w - x) dx.$$

The pdf of W (f_W) is called the **convolution** of f_X and f_Y.

Example

Let X, Y be independent, uniformly distributed random variables $X, Y \sim \mathcal{U}[0,1]$. Determine the pdf of $W = X + Y$.

Solution

$$S_W = [0,2],$$

$$F_W(w) = \mathbb{P}(W \leq w) = \mathbb{P}(X + Y \leq w)$$
$$= \begin{cases} 0, & w < 0, \\ \int_0^w \int_0^{w-x} 1 dydx = \frac{w^2}{2}, & 0 \leq w < 1, \\ 1 - \int_{w-1}^1 \int_{w-x}^1 1 dydx = -1 + 2w - \frac{w^2}{2}, & 1 \leq w < 2, \\ 1, & w \geq 2. \end{cases}$$